THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 1

1. Suppose that $\mathbf{u} = 5\mathbf{i} + 12\mathbf{j}$ and $\mathbf{v} = -6\mathbf{i} - 8\mathbf{j}$. Find

   (a) $\mathbf{u} \cdot \mathbf{v}$,

   (b) $|\mathbf{u}|$ and $|\mathbf{v}|$,

   (c) the angle between $\mathbf{u}$ and $\mathbf{v}$.

   Ans:

   (a) $\mathbf{u} \cdot \mathbf{v} = (5)(-6) + (12)(-8) = -126$

   (b) $|\mathbf{u}| = \sqrt{5^2 + 12^2} = 13$ and $|\mathbf{v}| = \sqrt{(-6)^2 + (-8)^2} = 10$.

   (c) The angle between $\mathbf{u}$ and $\mathbf{v}$ $= \cos^{-1}\left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(-\dfrac{63}{65}\right) \approx 166°$.

2. Let $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

   (a) Express $\mathbf{a} = \mathbf{u} + \mathbf{v}$ such that $\mathbf{u}$ is parallel to $\mathbf{b}$ and $\mathbf{v}$ is orthogonal to $\mathbf{b}$.

   (b) Find the area of parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

   Ans:

   (a) We have
$$\mathbf{u} = \mathrm{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \frac{8}{9}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$
   and
$$\mathbf{v} = \mathbf{a} - \mathbf{u} = \frac{1}{9}(28\mathbf{i} - 43\mathbf{j} + 29\mathbf{j}).$$

   (b) The area of parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is
$$|\mathbf{a} \times \mathbf{b}| = |\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 5 \\ 1 & 2 & 2 \end{vmatrix}| = | - 16\mathbf{i} - 3\mathbf{j} + 11\mathbf{k}| = \sqrt{386}.$$

3. Let $A = (3, 3, 0)$, $B = (-2, -3, 2)$ and $C = (1, 0, 3)$ be three points in $\mathbb{R}^3$. Find the volume of the tetrahedron $OABC$.

   Ans:

$$\begin{aligned} \text{volume of the tetrahedron } OABC &= \frac{1}{6} \times \text{volume of the parallelepipe spanned by } \overrightarrow{OA}, \overrightarrow{OB} \text{ and } \overrightarrow{OC} \\ &= \frac{1}{6} \times |\begin{vmatrix} 3 & 3 & 0 \\ -2 & -3 & 2 \\ 1 & 0 & 3 \end{vmatrix}| \\ &= \frac{1}{6} \times | - 3| \\ &= \frac{1}{2} \end{aligned}$$

4. Let A and B be two points in $\mathbb{R}^n$ and let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

Suppose that C is a point on AB such that $AC : CB = r : s$, where $r, s \in \mathbb{R}$. Show that

$$\overrightarrow{OC} = \frac{1}{r+s}(r\mathbf{b} + s\mathbf{a}).$$

Ans:

Note that $\overrightarrow{AC} = \frac{r}{r+s}\overrightarrow{AB} = \frac{r}{r+s}(\mathbf{b} - \mathbf{a})$. Then,

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \frac{r}{r+s}(\mathbf{b} - \mathbf{a}) = \frac{1}{r+s}(r\mathbf{b} + s\mathbf{a}).$$

Remark: In particular, if $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are two points in $\mathbb{R}^2$ (i.e. $\overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\overrightarrow{OB} = x_2\mathbf{i} + y_2\mathbf{j}$), then we have $\overrightarrow{OC} = (\frac{rx_2 + sx_1}{r+s})\mathbf{i} + (\frac{ry_2 + sy_1}{r+s})\mathbf{j}$, that means $C = (\frac{rx_2 + sx_1}{r+s}, \frac{ry_2 + sy_1}{r+s})$, which is the section formula we learned in secondary school.

5. Let $\mathbf{p}$ and $\mathbf{q}$ be nonzero vectors in $\mathbb{R}^n$ such that they are not parallel and let $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Prove that if $a_1\mathbf{p} + a_2\mathbf{q} = b_1\mathbf{p} + b_2\mathbf{q}$, then $a_1 = b_1$ and $a_2 = b_2$.

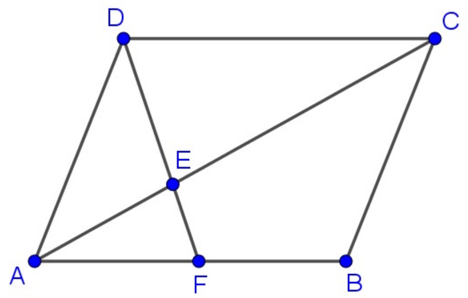
Ans:

If $a_1\mathbf{p} + a_2\mathbf{q} = b_1\mathbf{p} + b_2\mathbf{q}$, then we have $(a_1 - b_1)\mathbf{p} = (b_2 - a_2)\mathbf{q}$.

We claim that $a_1 = b_1$. Otherwise, $a_1 - b_1 \neq 0$ and we have $\mathbf{p} = \frac{b_2 - a_2}{a_1 - b_1}\mathbf{q}$ which implies $\mathbf{p}$ is parallel to $\mathbf{q}$ that contradicts to the assumption. Therefore, we have $a_1 = b_1$.

As a result $a_1 - b_1 = 0$. Therefore, $(b_2 - a_2)\mathbf{q} = \mathbf{0}$, but $\mathbf{q}$ is nonzero, so $b_2 - a_2 = 0$, i.e. $a_2 = b_2$.

6.



In the above diagram, $ABCD$ is a parallelogram and F is a point on AB.

Suppose that DF and AC intersect at the point E such that $DE : EF = \lambda : 1$, where $\lambda > 0$.

Let $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AD} = \mathbf{q}$, $\overrightarrow{AE} = h\overrightarrow{AC}$ and $\overrightarrow{AF} = k\overrightarrow{AB}$, where $h, k > 0$.

(a) i. Express $\overrightarrow{AE}$ in terms of h, $\mathbf{p}$ and $\mathbf{q}$.

ii. Express $\overrightarrow{AE}$ in terms of λ, k $\mathbf{p}$ and $\mathbf{q}$.
Hence, show that $\lambda = \frac{1}{k}$.

(b) Given that $|\mathbf{p}| = 3$, $|\mathbf{q}| = 2$ and $\angle DAB = \frac{\pi}{3}$.

i. Find $\mathbf{p} \cdot \mathbf{q}$.

ii. Suppose that DF is perpendicular to AC.

(1) Express $\overrightarrow{DF}$ in terms of k, $\mathbf{p}$ and $\mathbf{q}$, and so find the value of k.

(2) Using (a), find the length of AE.

Ans:

(a) i. $\overrightarrow{AE} = h\overrightarrow{AC} = h(\mathbf{p} + \mathbf{q})$

ii. $\overrightarrow{AE} = \frac{1}{1+\lambda}(\lambda\overrightarrow{AF} + 1\overrightarrow{AD}) = \frac{1}{1+\lambda}(k\lambda\mathbf{p} + \mathbf{q})$.

Therefore, $h(\mathbf{p}+\mathbf{q}) = \frac{1}{1+\lambda}(k\lambda\mathbf{p}+\mathbf{q})$. Then, we have $h = \frac{k\lambda}{1+\lambda}$ and $h = \frac{1}{1+\lambda}$.

Hence, $k\lambda = 1$ and the result follows.

(b) i. $\mathbf{p}\cdot\mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\angle DAB = 3$

ii. (1) $\overrightarrow{DF} = \overrightarrow{AF} - \overrightarrow{AD} = k\mathbf{p} - \mathbf{q}$.

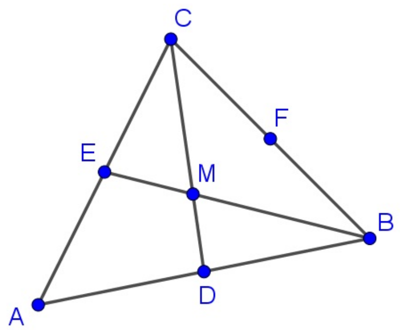
Since $\overrightarrow{DF} \perp \overrightarrow{AC}$, we have $\overrightarrow{DF}\cdot\overrightarrow{AC} = 0$. Then,

$$\begin{aligned} (k\mathbf{p}-\mathbf{q})\cdot(\mathbf{p}+\mathbf{q}) &= 0 \\ k|\mathbf{p}|^2 + (k-1)\mathbf{p}\cdot\mathbf{q} - |\mathbf{q}|^2 &= 0 \\ 9k + 3(k-1) - 4 &= 0 \\ k &= \frac{7}{12} \end{aligned}$$

(2) For $k = \frac{7}{12}$, $\lambda = \frac{12}{7}$, $h = \frac{7}{19}$. Then, $|\overrightarrow{AE}|^2 = |\frac{7}{19}(\mathbf{p}+\mathbf{q})|^2 = (\frac{7}{19})^2(\mathbf{p}+\mathbf{q})\cdot(\mathbf{p}+\mathbf{q}) = \frac{49}{19}$.

Hence, $|\overrightarrow{AE}| = \frac{7\sqrt{19}}{19}$.

7.



In the above diagram, A, B, C are three distinct points in $\mathbb{R}^2$ and let $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AC} = \mathbf{q}$.

Suppose that D, E and F are mid-points of AB, AC and BC respectively, M is the intersection of CD and BE.

(a) Suppose that $CM : MD = r : 1$ and $BM : ME = s : 1$, where $r, s > 0$.

i. Express $\overrightarrow{AM}$ in terms of r, $\mathbf{p}$ and $\mathbf{q}$.

ii. Express $\overrightarrow{AM}$ in terms of s, $\mathbf{p}$ and $\mathbf{q}$.

iii. Hence, show that $r = s = 2$ and $\overrightarrow{AM} = \frac{1}{3}(\mathbf{p}+\mathbf{q})$.

(b) Prove that three medians AF, BE and CD of $\triangle ABC$ intersect at the point M.

Also, prove that $CM : MD = BM : ME = AM : MF = 2 : 1$.

Ans:

(a) i. $\overrightarrow{AM} = \frac{1}{1+r}(1\overrightarrow{AC} + r\overrightarrow{AD}) = \frac{1}{1+r}(\mathbf{q} + \frac{r}{2}\mathbf{p}) = \frac{r}{2(1+r)}\mathbf{p} + \frac{1}{1+r}\mathbf{q}$

ii. $\overrightarrow{AM} = \frac{1}{1+s}(1\overrightarrow{AB} + s\overrightarrow{AE}) = \frac{1}{1+s}(\mathbf{p} + \frac{s}{2}\mathbf{q}) = \frac{1}{1+s}\mathbf{p} + \frac{s}{2(1+s)}\mathbf{q}$

iii. Hence, we have $\frac{r}{2(1+r)} = \frac{1}{1+s}$ and $\frac{1}{1+r} = \frac{s}{2(1+s)}$.

By solving the above, we have $r = s = 2$. By putting $r = 2$ into the result in (a)(i), we have $\overrightarrow{AM} = \frac{1}{3}(\mathbf{p}+\mathbf{q})$.

(b) Note that $\overrightarrow{AF} = \frac{1}{2}(\mathbf{p}+\mathbf{q})$, so $\overrightarrow{AM} = \frac{2}{3}\overrightarrow{AF}$ which means $\overrightarrow{AM}$ is parallel to $\overrightarrow{AF}$. Therefore, A, M and F are collinear.

Furthermore, $\overrightarrow{MF} = \frac{1}{3}\overrightarrow{AF}$, so $AM : MF = 2 : 1$. Therefore, we have $CM : MD = BM : ME = AM : MF = 2 : 1$.

8. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}$ and let

$$p(t) = \sum_{i=1}^{n} (a_i - b_i t)^2$$

be a polynomial.

By using the fact that $p(t) \geq 0$ for all $t \in \mathbb{R}$, prove that the Cauchy Schwarz inequality holds, i.e.

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

and the equality holds if and only if $a_1 = tb_1, a_2 = tb_2, \ldots, a_n = tb_n$ for some $t \in \mathbb{R}$.

Ans:

Note that

$$\begin{aligned} p(t) &= \sum_{i=1}^{n} (a_i - b_i t)^2 \\ &= \left(\sum_{i=1}^{n} b_i^2\right) t^2 - 2\left(\sum_{i=1}^{n} a_i b_i\right) t + \left(\sum_{i=1}^{n} a_i^2\right) t^2 \end{aligned}$$

Clearly, $\left(\sum_{i=1}^{n} b_i^2\right) \geq 0$. However, if $\left(\sum_{i=1}^{n} b_i^2\right) = 0$, then $b_1 = b_2 = \cdots = b_n = 0$ and so the required inequality is trivially true.

Now, we assume that $\left(\sum_{i=1}^{n} b_i^2\right) > 0$. Note that $p(t) \geq 0$ for all $t \in \mathbb{R}$, the quadratic equation $p(t) = 0$ has either no solution or only one solution. Therefore, the discriminant of $p(t)$ is less than or equals to 0.

$$\begin{aligned} \left[2\left(\sum_{i=1}^{n} a_i b_i\right)\right]^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1} b_i^2\right) &\leq 0 \\ 4\left(\sum_{i=1}^{n} a_i b_i\right)^2 &\leq 4\left(\sum_{i=1^n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \\ \left(\sum_{i=1}^{n} a_i b_i\right)^2 &\leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \end{aligned}$$

Furthermore, the equality holds if and only if the discriminant of $p(t)$ equals to 0, which means $p(t) = 0$ has exactly one solution. Therefore, each square in the summation of $p(t)$ must be zero, i.e. $a_i - tb_i = 0$, i.e. $a_i = tb_i$ for all $i = 1, 2, \ldots, n$.